



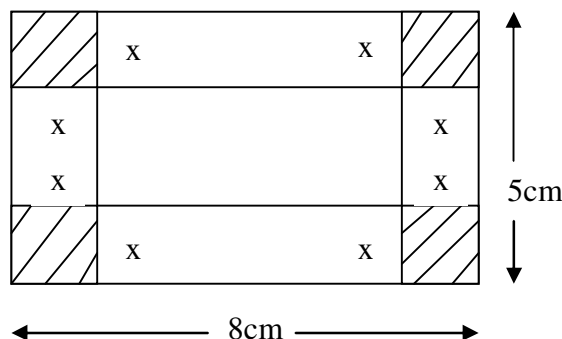
Pdf SMA 104 Lecture 12(Maximum Minimum, Curve Sketching)

Project Management (University of Nairobi)

APPLICATIONS OF MAXIMUM AND MINIMUM VALUES

Examples:

1. The figure below represents a rectangular sheet of metal measuring 8cm by 5cm. Equal squares of side x cm are removed from each corner, and the edges are then turned up to make an open box of volume v cm³. Show that $v = 40x - 26x^2 + 4x^3$. Hence find the maximum possible volume and the corresponding value for x .



Solution: Dimensions of the cube are:-

Length = $(8 - 2x)$ cm ; Width = $(5 - 2x)$ cm ; height = x cm

$$\therefore v = l \times w \times h = (8 - 2x)(5 - 2x)x \text{ cm}^3$$

$$= 40x - 26x^2 + 4x^3$$

Finding maximum possible volume

$$\text{For maximum or minimum volume, } \frac{dv}{dx} = 0 \Rightarrow 40 - 52x + 12x^2 = 0$$

$$\Rightarrow 3x^2 - 13x + 10 = 0$$

$$\Rightarrow x = 1 \text{ or } x = \frac{10}{3}$$

$$\text{When, } x = 1, \frac{d^2v}{dx^2} = -52 + 24x$$

$$= -52 + 24 = -28 < 0$$

\therefore Volume is maximum, and the value of the maximum volume is

$$v = 40(1) - 26(1)^2 + 4(1)^3 = 18 \text{ cm}^3$$

$$\text{When } x = \frac{10}{3}, \frac{d^2v}{dx^2} = -52 + 24x$$

$$= -52 + 24\left(\frac{10}{3}\right) = -52 + 80$$

$$= 28 > 0$$

\therefore Volume is minimum

The value of this minimum volume (ALTHOUGH NOT ASKED) is

$$40\left(\frac{10}{3}\right) - 26\left(\frac{10}{3}\right)^2 + 4\left(\frac{10}{3}\right)^3 \approx 7.41 \text{ cm}^3$$

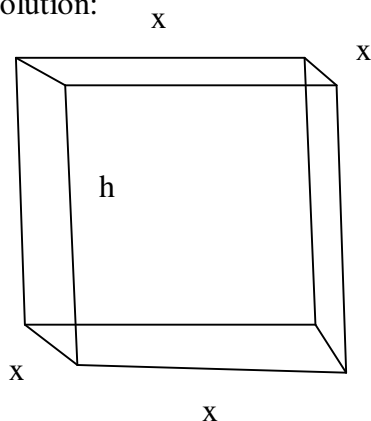
REQUIRED ANSWER: the maximum volume is 18cm^3 when $x = 1$

(2) Two opposite ends of a closed rectangular tank are squares of sides $x\text{cm}$ and the total area of the sheet of metal forming the tank is $s\text{m}^2$.

(a) Show that the volume of the tank is $\frac{1}{4}x(5 - 2x^2)\text{m}^3$.

(b) If the value of $s = 2400$, find the value of x for which the volume is a maximum.

Solution:



(a) Surface area $= 2x^2 + 4(xh) = s$

$$2x^2 + 4xh = s$$

$$h = \frac{1}{4} \left(\frac{s - 2x^2}{x} \right)$$

$$v = l \times w \times h = x \times x \times \frac{1}{4} \left(\frac{s - 2x^2}{x} \right)$$

$$= \frac{1}{4} x (s - 2x^2)$$

(b) $v = \frac{1}{4} x (2400 - 2x^2)$

$$v = 600x - \frac{1}{2}x^3$$

For maximum volume or minimum value,

$$\frac{dv}{dx} = 600 - \frac{3x^2}{2} = 0; x^2 = 400; x = \pm 20$$

When $x = 20$, $v = 600(20) - \frac{1}{2}(20)^3 = 8000\text{cm}^3$

When $x = -20$, $v = 600(-20) - \frac{1}{2}(-20)^3$

$$= -12000 + 4000$$

$$= -8000$$

Check if maximum or minimum $\frac{d^2v}{dx^2} = \frac{-6x}{2} = -3x$

When $x = 20$, $\frac{d^2v}{dx^2} = -3(20) = -60 < 0$

$\therefore v$ is maximum

When $x = -20$, $\frac{d^2v}{dx^2} = -3(-20) = 60 > 0$

$\therefore v$ is minimum(not required)

Required answer:

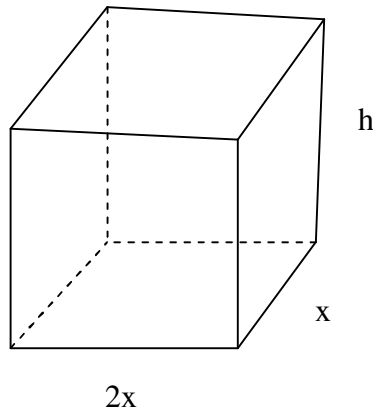
The volume is a maximum when $x = 20$, and its value is 8000cm^3 .

(3) The length of a rectangular block is twice the width and the total surface area is

108cm^2 . Show that if the width of the block is $x\text{cm}$, the volume is $\frac{4}{3}x(27 - x^2)\text{cm}^3$.

Find the dimensions of the block when its volume is a maximum.

Solution:



$$h = \frac{108 - 4x^2}{6x} \quad \left(\text{since } \begin{matrix} s = 4xh + 2xh + 4x^2 \\ 108 = 4xh + 2xh + 4x^2 \end{matrix} \right)$$

$$v = l \times w \times h = \left(\frac{108 - 4x^2}{6x} \right) (2x)(x) \text{cm}^3$$

$$= \frac{1}{3}x(108 - 4x^2) \text{cm}^3$$

$$v = \frac{4}{3}x(27 - x^2) \text{cm}^3$$

For maximum volume or minimum volume,

$$\frac{dv}{dx} = 0;$$

$$v = 36x - \frac{4}{3}x^3$$

$$\frac{dv}{dx} = 36 - \frac{4}{3} \cdot 3x^2; 36 - 4x^2 = 0$$

$$x = \pm 3$$

Check if maximum or minimum $\frac{d^2v}{dx^2} = -8x$

$$\text{When } x = 3, \frac{d^2v}{dx^2} = -8(3) = -24 < 0$$

$\Rightarrow v$ is maximum

$$\text{When } x = -3, \frac{d^2v}{dx^2} = -8(-3) = 24 > 0$$

$\Rightarrow v$ is minimum

\therefore maximum volume occurs when $x = 3$, and in this case $l = 6$, $w = 3$, $h = 4$

Curve sketching

The determination of maximum, minimum points and points of inflection is important in sketching a curve.

Example:

Sketch the curve of $y = 4x^3 - 3x^4$.

Solution:

Step 1: Find the point where the curve meets the x -axis as follows:

At the x -intercept, $y = 0 \Rightarrow$

$$4x^3 - 3x^4 = 0 \Rightarrow x^3(4 - 3x) = 0$$

$$\Rightarrow x = 0 \text{ or } x = \frac{4}{3}$$

Get the coordinates of these points

$$\text{When } x = 0, y = 4(0)^3 - 3(0)^4 = 0$$

$$\text{When } x = \frac{4}{3}, y = 4\left(\frac{4}{3}\right)^3 - 3\left(\frac{4}{3}\right)^4 = 0$$

\therefore The curve meets the x -axis at the points $(0, 0)$ and $(\frac{4}{3}, 0)$.

Step 2: Find the point where the curve meets the y -axis as follows:

$$\text{At the } y\text{-axis, } x = 0 \Rightarrow y = 4(0)^3 - 3(0)^3 = 0$$

\therefore the y -intercept is $(0, 0)$

Step 3: Find the stationary/Turning points

Stationary/Turning points occur when $\frac{dy}{dx} = 0$

$$\text{Now, } \frac{dy}{dx} = 0 \Rightarrow 12x^2 - 12x^3 = 0 \Rightarrow 12x^2(1 - x) = 0$$

$$\Rightarrow x = 0 \text{ or } x = 1$$

When $x = 0, y = 0$ and when $x = 1, y = 1$

\therefore Turning points are at $(0, 0)$ and $(1, 1)$

Step 4: Determine the nature of the turning point

Using the second derivative test,

$$\frac{d^2y}{dx^2} = 24x - 36x^2; \text{ when } x = 0,$$

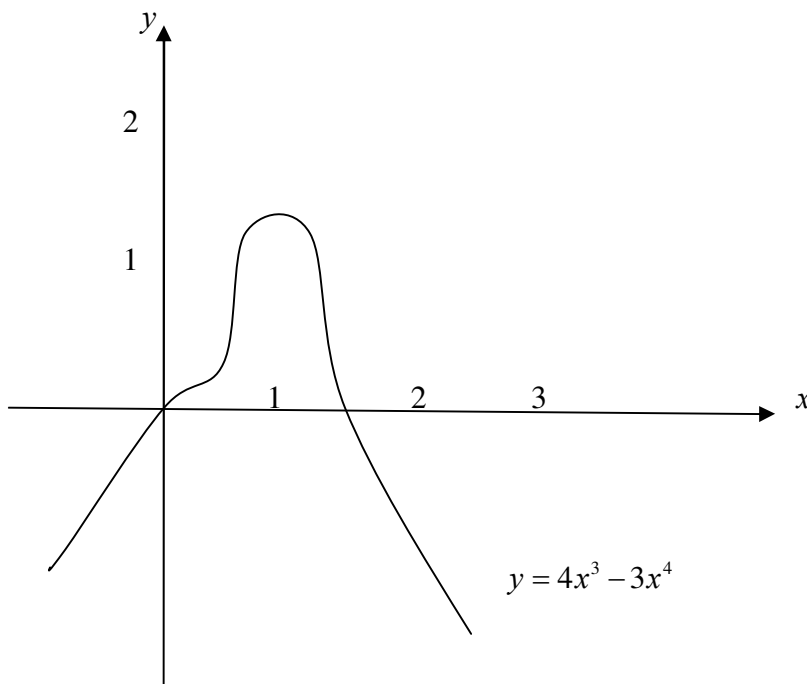
$$\frac{d^2y}{dx^2} = 0 \Rightarrow (0,0) \text{ is a point of inflection}$$

$$\text{At } x = 1, \frac{d^2y}{dx^2} = 24(1) - 36(1) = -12 < 0 \Rightarrow$$

$(1,1)$ is a maximum point.

Confirm these results using the first derivative test

Sketching



Exercise

1. Sketch the following curves

$$(1) y = 3x^2 - x^3 \quad (2) y = x^3 - 2x^2 + x \quad (3) y = (x+1)(x-1)(2-x)$$

Asymptotes

Asymptote: An asymptote is a straight line which the curve being studied approaches. Alternatively, an asymptote is a straight line to which the curve $y = f(x)$ approaches closer and closer as one moves along it. The asymptotes are vertical, horizontal and slant/oblique asymptotes.

1. Vertical asymptotes

These correspond to the zeroes of the denominator of a fraction.

Examples

(a) Find vertical asymptotes of $y = \frac{x^2 + 2x - 3}{x^2 - 5x - 6}$;

Solution: The denominator is $x^2 - 5x - 6$; Zero of denominator $x^2 - 5x - 6 = 0$.

$$(x-6)(x+1)=0; x=6 \text{ or } x=-1$$

So x cannot be 6 or -1 because in this case the denominator will be zero and dividing by zero will give an undefined value.

Therefore $x=6$ and $x=-1$ are the vertical asymptotes.

(b) Find vertical asymptotes of $y = \frac{x+2}{x^2+2x-8}$

Solution: The denominator is $x^2+2x-8=0$. Zero of denominator $x^2+2x-8=0$

$$(x+4)(x-2)=0; \quad x=-4, \quad x=2$$

2. Horizontal asymptotes

The line $y=b$ is a horizontal asymptote for $y=f(x)$ if $\lim_{x \rightarrow +\infty} f(x)=b$ or $\lim_{x \rightarrow -\infty} f(x)=b$

Examples

(a) Find Horizontal asymptotes of $y = \frac{x^2+2x-3}{x^2-5x-6}$

Solution:

$$\lim_{x \rightarrow \pm\infty} \frac{x^2+2x-3}{x^2-5x-6} = \lim_{x \rightarrow \pm\infty} \frac{1+\frac{2}{x}-\frac{3}{x^2}}{1-\frac{5}{x}-\frac{6}{x^2}} = 1; \quad \therefore y=1 \text{ is the horizontal asymptote.}$$

(b) Find Horizontal asymptotes of $y = \frac{x+2}{x-1}$

Solution:

$$\lim_{x \rightarrow \infty} \frac{x+2}{x-1} = \lim_{x \rightarrow \infty} \frac{1+\frac{2}{x}}{1-\frac{1}{x}} = 1 \quad \therefore y=1 \text{ is the horizontal asymptote.}$$

Or make x the subject of the formula; $(x-1)y = x+2$; $xy - y = x+2$; $xy - x = y+2$;

$$x(y-1) = y+2$$

$$x = \frac{y+2}{y-1} \quad \text{and equate } y-1=0; \quad y=1 \text{ which is the asymptote.}$$

(c). Find vertical and horizontal asymptotes, if any, of $y = \frac{2x}{x+1}$ and sketch the curve.

Solution:

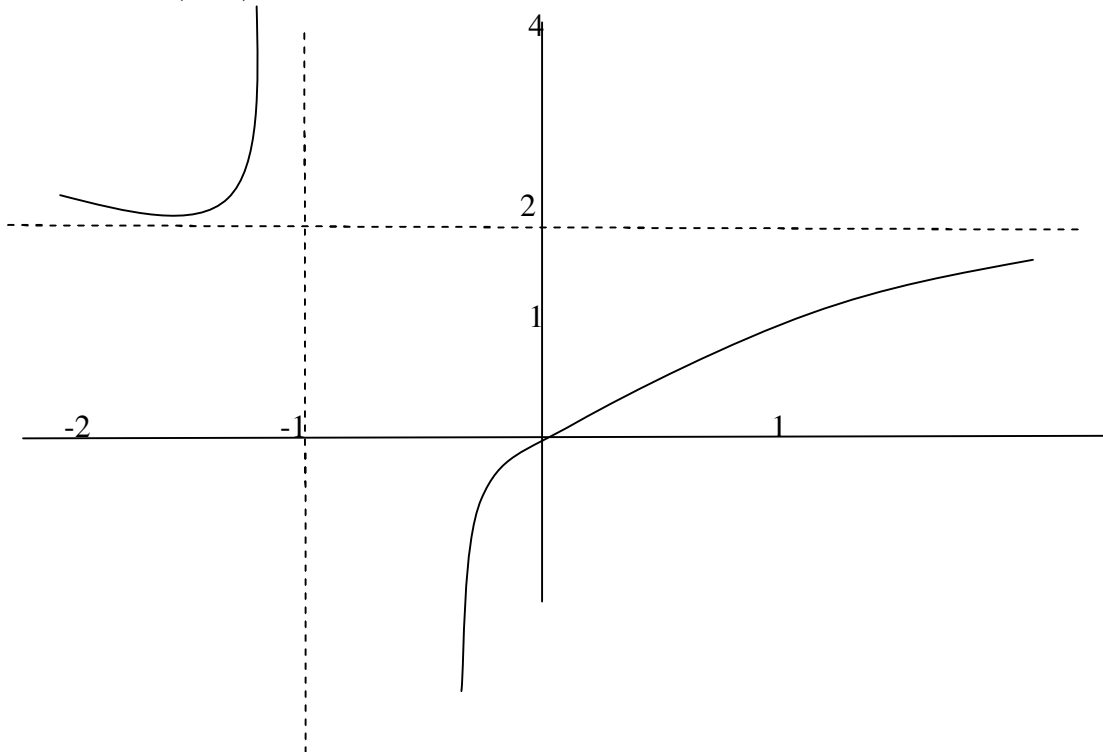
Vertical asymptote: $x+1=0$; $x=-1$ is the vertical asymptote

Horizontal asymptote: $\lim_{x \rightarrow \pm\infty} \frac{2x}{x+1} = \lim_{x \rightarrow \pm\infty} \frac{2}{1+\frac{1}{x}} = 2$; $\therefore y=2$ is the horizontal asymptote

x -intercept, $y=0$, $x=0$; y -intercept, $x=0$, $y=0$

Turning points

$$\frac{dy}{dx} = \frac{2(x+1)-2x}{(x+1)^2}; \quad \frac{dx}{dy} = \frac{(x+1)^2}{2} = 0; \quad (x+1)^2 = 0; \quad x = -1; \quad y = \infty$$



3. Oblique or slant asymptote

If $y = \frac{h(x)}{g(x)}$ where the degree of $h(x)$ minus the degree of $g(x)$ equals to one, then $y = f(x)$ has

an oblique asymptote of the form $y = mx + c$ where $y = f(x) = mx + c + \frac{a}{g(x)}$

Example

(a) Find all the asymptotes, if any, of $y = \frac{x^2 - 4}{x - 1}$ and sketch the curve.

Solution: Vertical asymptote: $x - 1 = 0$; $x = 1$

Horizontal asymptote: $\lim_{x \rightarrow \pm\infty} \frac{x^2 - 4}{x - 1} = \lim_{x \rightarrow \pm\infty} \frac{1 - \frac{4}{x^2}}{\frac{1}{x} - \frac{1}{x^2}} = \infty$; No limit hence no horizontal asymptote.

Oblique asymptote: We find by long division

$$\begin{array}{r} x+1 \\ x-1 \overline{) x^2-4} \\ \underline{x^2-x} \\ x-4 \\ \underline{x-1} \\ -3 \end{array} \quad y = \frac{x^2 - 4}{x - 1} = (x + 1) + \frac{-3}{x - 1}; \therefore y = x + 1$$

$\therefore y = x + 1$ is the oblique asymptote

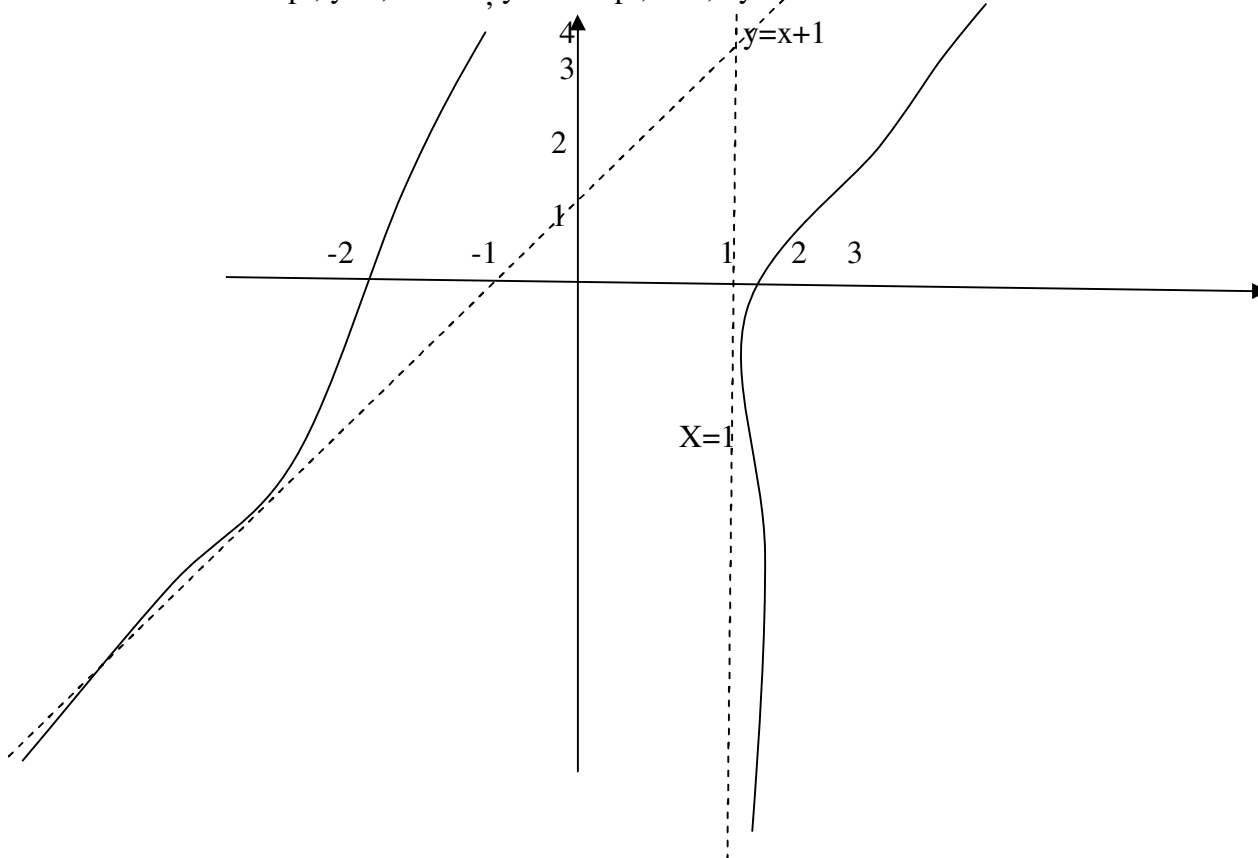
Turning points:

$$y = \frac{x^2 - 4}{x - 1}; \quad \frac{dy}{dx} = \frac{(x-1)2x - (x^2 - 4)}{(x-1)^2} = 0; \quad = \frac{2x^2 - 2x - x^2 + 4}{(x-1)^2} = 0; \quad \frac{dy}{dx} = \frac{x^2 - 2x + 4}{(x-1)^2} = 0;$$

$$y' = x^2 - 2x + 4 = 0$$

$$x = \frac{+2 \pm \sqrt{4 - 16}}{2} \text{ no real roots so no turning points.}$$

x-intercept, $y=0$; $x \pm 2$; y-intercept, $x=0$, $y=4$.



(b) Determine the vertical, horizontal and oblique asymptotes if any of the function

$$y = f(x) = \frac{x^2 + 3x + 6}{x - 4}. \text{ Hence or otherwise sketch the graph.}$$

Solution:

Vertical asymptote: $x - 4 = 0$, $x = 4$

Horizontal asymptote: $\lim_{x \rightarrow \pm\infty} \frac{x^2 + 3x + 6}{x - 4} = \frac{1 + \frac{3}{x} + \frac{6}{x^2}}{\frac{1}{x} - \frac{4}{x^2}} = \infty$ (undefined), therefore there is no

horizontal asymptote.

Oblique asymptote $x - 4 \overline{) \begin{array}{r} x + 7 \\ x^2 + 3x + 6 \\ \underline{x^2 - 4x} \\ 7x + 6 \\ \underline{7x - 28} \\ 34 \end{array}}$; $y = (x + 7) + \frac{34}{x - 4}$; $y = x + 7$ is the oblique asymptote

Turning points: $\frac{dy}{dx} = \frac{(2x + 3)(x - 4) - (x^2 + 3x + 6)}{(x - 4)^2} = 0$; $= \frac{x^2 - 8x - 18}{(x - 4)^2} = 0$; $x^2 - 8x - 18 = 0$.

$$x = \frac{8 \pm \sqrt{64 + 72}}{2}; \quad x = 9.8; \quad x = -1.8$$

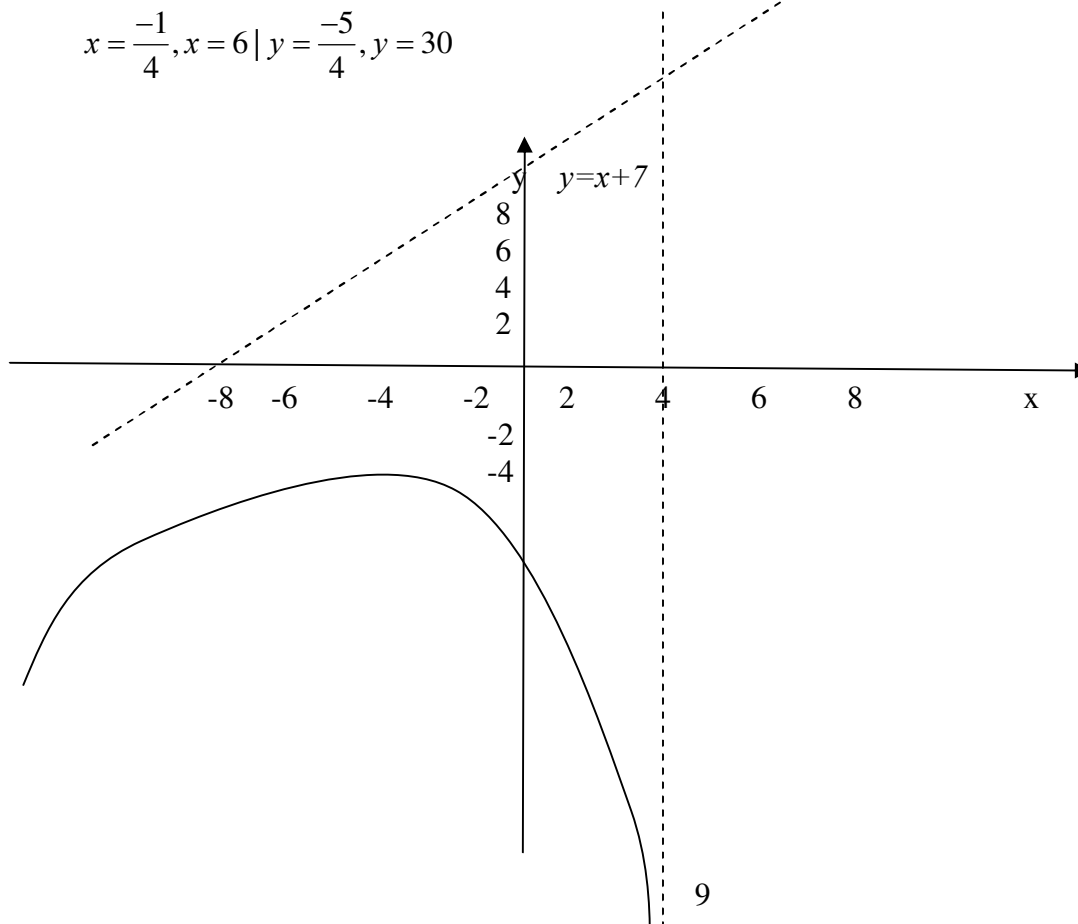
Or

$$y = \frac{x^2 + 3x + 6}{x - 4}; \quad \frac{dy}{dx} = \frac{(2x + 3)(x - 4) - (x^2 + 3x + 6)}{(x - 4)^2} = 0;$$

$$2x^2 - 8x + 3x^2 - 12x - x^2 - 3x - 6 = 0$$

$$4x^2 - 23x - 6 = 0; \quad 4x^2 - 24x + x - 6 = 0; \quad 4x(x - 6) + 1(x - 6) = 0;$$

$$x = \frac{-1}{4}, x = 6 \mid y = \frac{-5}{4}, y = 30$$



In finding the asymptotes, observe the following:

1. To get vertical asymptotes, set the determinant equal to zero and solve for the zeros (if any).
2. Compare the degrees of the numerator and the denominator.

If the degrees are the same, then you have a horizontal asymptote.

If the degree of the denominator is greater than that of the numerator, then you have a horizontal asymptote at $y=0$

If the numerators degree is greater (by a margin 1), then you have a slant or oblique asymptote which you will find by long division.

SUMMARY OF CURVE SKETCHING

1. Determine the y and x intercepts
2. Determine the asymptotes of the graph if any.
3. Determine the turning points and distinguish them (i.e. check maximum, minimum and point of inflection).
4. Hence sketch the points.

Note:

A curve will have vertical asymptotes if, when its equation is written in the form

$$ay^n + (bx + c)y^{n-1} + (dx^2 + ex + f)y^{n-2} + \dots U_n(x) = 0 \dots (1)$$

Where $U_n(x)$ is a polynomial in x of degree n .

The coefficient of the highest power of y is a non-constant function of x having one or more (real) linear factors.

To each such factor, there corresponds a vertical asymptote.

A curve will have horizontal asymptotes if, when its equation is written in the

form $ax^n + (by + c)x^{n-1} + (dy^2 + ey + f)x^{n-2} + \dots = 0$, the coefficient of the highest power of x is a non-constant function of y having one or more (real) linear factors. To such factors, there corresponds a horizontal asymptote.

To obtain the equations of the oblique asymptotes

1. Replace y by $mx + b$ in the equation of the curve and arrange the result in the form

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n = 0 \dots (3)$$

2. Solve simultaneously the equations $a_0 = 0$ and $a_1 = 0$ for m and b .

For each pair of solutions m and b , write the equation of an asymptote $y = mx + b$.

3. If $a_1 = 0$, irrespective of the value of b , the equations $a_0 = 0$ and $a_2 = 0$ are to be used in (3).

Exercise: Find the equations of the asymptotes of $y^2(1+x) = x^2(1-x)$.