Pdf SMA 104 Lecture 2(Limits AND Continuity)

Project Management (University of Nairobi)

Limits

The concept of limits of a function is one of the fundamental ideas that distinguishes Calculus from other areas of mathematics e.g. Algebra or Geometry.

If *f(x)* becomes arbitrarily close to a single number *L* as *x* approaches *a* from either side, then the limit of *f(x)* as *x* approaches *a* is *L* written as $\lim_{x \to a} f(x) = L$.

Consider a function *y=f(x)*

$\lim_{x \to a} f(x) = L$ means the limit of *f(x)* as *x* approaches *a* is equal to a number *L* i.e. as *x* gets closer and closer to a ($x \neq a$), *f(x)* gets closer and closer to *L*.

Example 21: Let $f(x) = x^2$. Find $\lim_{x \to 2} f(x)$

Solution:

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} x^2 = 2^2 = 4$$

Example 22: Let *f(x) = 5x – 3*. Find $\lim_{x \to 2} 5x - 3$

Solution:

$$\lim_{x \to 2} 5x - 3 = (5 \times 2 - 3) = 7$$

Example 23: Let

$$f(x) = \frac{1}{x}.$$

Find

$$\lim_{x \to \infty} \frac{1}{x}$$

Solution:

$$\lim_{x \to 0} \frac{1}{x} = \infty \quad (undefined)$$

Properties of limits

1. $\lim_{x \to a} k = k$
2. $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
3. $\lim_{x \to a} f(x) \times g(x) = \lim_{x \to a} f(x) \times \lim_{x \to a} g(x)$

4. $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ *provided that* $\lim g(x)\neq 0$

5. $\lim \sqrt[n]{f(x)}=\sqrt[n]{\lim_{x\to a} f(x)}$

$e.g \quad \lim_{x\to a} x^{\frac{1}{2}}=\left(\lim_{x\to a} x\right)^{\frac{1}{2}}$

Example 24:

$$\lim_{x\to 5}x^2-4x+3=\lim_{x\to 5}x^2-\lim_{x\to 5}4x+\lim_{x\to 5}3$$

$$=5^2-4\times 5+3$$
$$=25-20+3$$
$$=8$$

Example 25:

$$\lim_{x\to 2}\frac{3x+5}{5x+7}=\frac{\lim_{x\to 2}3x+5}{\lim_{x\to 2}5x+7}=\frac{3\times 2+5}{5\times 2+7}=\frac{11}{17}$$

Example 26:

$$\lim_{x\to 2}\frac{x^2-4}{x-2}\neq\frac{\lim_{x\to 2}x^2-4}{\lim_{x\to 2}x-2} \quad \text{since} \lim_{x\to 2}x-2=0$$

Hence $\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{(x+2)(x-2)}{(x-2)}=\lim_{x\to 2}(x+2)=4$

Example 27:

$$\lim_{x\to 8}\frac{x^{\frac{2}{3}}+3\sqrt{x}}{4-\frac{16}{x}}=\frac{\lim_{x\to 8}x^{\frac{2}{3}}+\lim_{x\to 8}3\sqrt{x}}{\lim_{x\to 8}4-\lim_{x\to 8}\frac{16}{x}}$$

$$=\frac{8^{\frac{2}{3}}+3\sqrt{8}}{4-\frac{16}{8}}$$

$$=\frac{4+6\sqrt{2}}{2}$$

$$=2+3\sqrt{2}$$

Example 28:

$$\lim_{x \to \infty} \frac{3x + 5}{6x - 8} = \lim_{x \to \infty} \frac{\frac{3x}{x} + \frac{5}{x}}{6 - \frac{8}{x}} = \lim_{x \to \infty} \frac{3 + \frac{5}{x}}{6 - \frac{8}{x}}$$

$$\frac{\lim_{x \to \infty} 3 + \lim_{x \to \infty} \frac{5}{x}}{\lim_{x \to \infty} 6 - \lim_{x \to \infty} 8} = \frac{3 + 0}{6 - 0} = \frac{1}{2}$$

Example 29: $\lim_{x \to \infty} \frac{4x^2 - x}{2x^3 - 5}$ Divide by the highest power of x.

$$\lim_{x \to \infty} \left(\frac{\frac{4}{x} - \frac{1}{x^2}}{2 - \frac{5}{x^3}} \right) = \frac{0 - 0}{2 - 0} = \frac{0}{2} = 0$$

Example 30:

$$\lim_{x \to \infty} \frac{\sqrt{x^2 + 2}}{3x - 6} = \frac{\lim_{x \to \infty} \sqrt{x^2 \left(1 + \frac{2}{x^2}\right)}}{3x - 6}$$

$$= \lim_{x \to \infty} \frac{x\left(1 + \frac{2}{x^2}\right)^{\frac{1}{2}}}{3x - 6}$$

$$= \lim \frac{x\sqrt{\left(1 + \frac{2}{x^2}\right)}}{x\left(3 - \frac{6}{x}\right)}$$

$$= \frac{\lim_{x \to \infty} \sqrt{1 + \frac{2}{x^2}}}{\lim_{x \to \infty} \left(3 - \frac{6}{x}\right)}$$

$$= \frac{1}{3}$$

Example 31:

$$\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = 3$$

$$\lim_{x \to 1} \frac{(x^2 + x + 1)(x - 1)}{(x - 1)} = \lim_{x \to 1} x^2 + x + 1$$

$$= 1 + 1 + 1$$

$$= 3$$

Example 32:

$$\lim_{x \to 2} \left(\frac{x^3 - 8}{x - 2} \right) = \frac{0}{0}$$

$$\lim_{x \to 2} \frac{(x - 2)(x^2 + 2x + 4)}{x - 2}$$

$$\lim_{x \to 2} x^2 + 2x + 4 = 4 + 4 + 4 = 12$$

Exercise 3

1. $\lim_{x \to \infty} \dfrac{5x + 1}{10 + 2x}$
2. $\lim_{x \to 5} \dfrac{x^2 - 4x - 5}{x - 5}$
3. $\lim_{x \to 5} \dfrac{x^2 - 25}{x - 5}$
4. $\lim_{x \to 0} \dfrac{\sqrt{2 + x} - \sqrt{2}}{x}$

5.The domain of the functions $f(x) = \dfrac{x}{5}$ and $g(x) = 7 - x$ is $\Re$ .Write down as simply as possible.

a. $f^{-1}(x)$ b. $g^{-1}(x)$ c. $fg(x)$ d. $(fg)^{-1}(x)$

Solutions to Exercise 3

1. $\lim_{x \to \infty} \dfrac{5x + 1}{10 + 2x} = \lim_{x \to \infty} \dfrac{5 + \frac{1}{x}}{\frac{10}{x} + 2} = 2\dfrac{1}{2}$

2. $\lim_{x \to 5} \frac{x^2 - 4x - 5}{x - 5} = \lim_{x \to 5} \frac{(x-5)(x+1)}{(x-5)}$

$= \lim_{x \to 5} x + 1 = 6$

Or $\lim_{x \to 5} \frac{2x - 4}{1} = 2(5) - 4 = 6$

3. $\lim_{x \to 5} \frac{\times 2 - 25}{x - 5} = \lim_{x \to 5} \frac{(x+5)(x-5)}{(x-5)} = 10$

Or $\lim_{x \to 5} \frac{2x}{1} = 2(5) = 10$

4. $\lim_{x \to 0} \frac{\sqrt{2-x} - \sqrt{2}}{x} \times \frac{\sqrt{2-x} + \sqrt{2}}{\sqrt{2} - x + \sqrt{2}}$

$= \frac{2 - x - 2}{x\left(\sqrt{2} + \sqrt{2-x}\right)}$

$= \frac{-x}{x\left(\sqrt{2} + \sqrt{2-x}\right)} = \frac{-1}{\sqrt{2} + \sqrt{2-x}}$

$\lim_{x \to 0} \frac{-1}{\sqrt{2} + \sqrt{2-x}} = \frac{-1}{\sqrt{2} + \sqrt{2}} = \frac{-1}{2\sqrt{2}} \times \frac{2\sqrt{2}}{2\sqrt{2}} = \frac{2\sqrt{2}}{4 \times 2} = \frac{\sqrt{2}}{4}$

$L^{'}$ Hospital Rule

$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0} \; or \; \infty$

Then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f^{'}(x)}{g^{'}(x)}$

e.g

1. $\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = \lim_{x \to 1} \frac{3x^2}{1}$

$= 3 \times 1$
$= 3$

2. $\lim_{x \to 2} \frac{x^3 - 8}{x - 2} = \frac{0}{0} \lim_{x \to 2} \frac{3x^2}{1} = 12$

3. $\lim_{x \to 0} \frac{\cos x - 2x - 1}{3x} = \lim_{x \to 0} \frac{-\sin x}{3} = \frac{-2}{3}$

Continuity

Continuity at a point.

A function is considered continous if the following conditions are met.

1. $f(a)$ is defined.
2. $\lim_{x\to a} f(x)$ exists.
3. $\lim_{x\to a} f(x) = f(a)$

Otherwise it is discontinuous.

Example 33: Show that $f(x)=\frac{x^2-4}{x-2}$ is not continous at x=2

Solution:

Condition 1: $f(2)=\frac{4-4}{2-2}=\frac{0}{0}$, which is undefined

Condition 2: $\lim_{x\to 2} f(x) = \lim_{x\to 2}\frac{x^2-4}{x-2}$

$= \lim_{x\to 2}\frac{(x+2)(x-2)}{x-2}$

$= \lim_{x\to 2} x+2 = 4$

Therefore, $\lim_{x\to 2} f(x)$ exists.

Condition 3: $\lim_{x\to 2} f(x) = 4$, but $f(2)$ is undefined

$\therefore \lim_{x\to 2} f(x) \neq f(2)$

Therefore f(x) is not continous at x=2

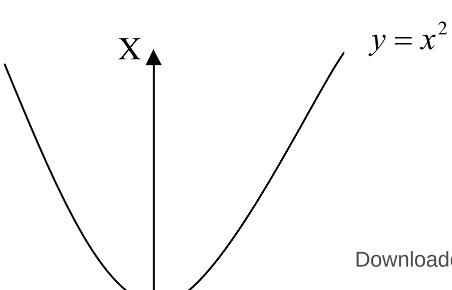
Note: It is possible to redefine f(x) to make it continous at x=2, as follows:

$$f(x)=\begin{cases}\frac{x^2-4}{x-2}, x\neq 2\\ 4, \quad x=2\end{cases}$$

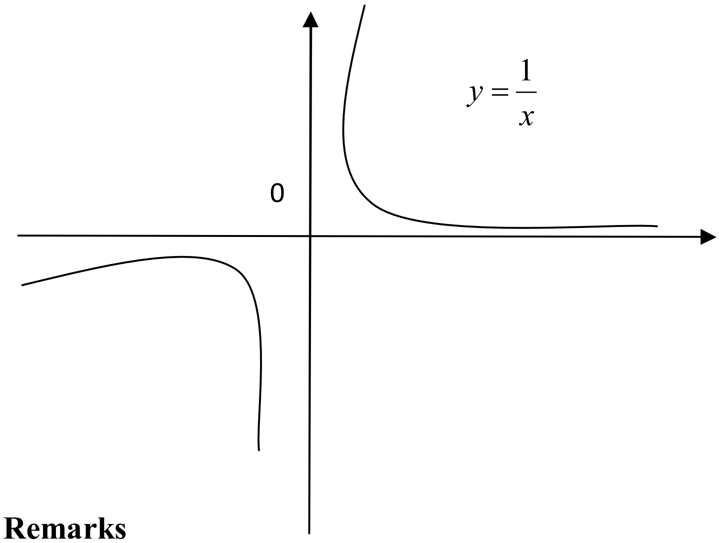
$\lim_{x\to 2} f(x)=4$, i.e. $\lim_{x\to 2}$ *exists*, we redefine f(x) so that

$\lim_{x\to 2} f(x) = f(2) = 4$

Example of a continous function.



Example of a discontinous function.



Remarks

1.Polynomials are always continous functions.

e.g $f(x)=x^2-2x+1$ at c since

Condition 1: $f(c)$ is defined i.e. $f(c)=c^2-2c+1$

Condition 2: $\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c} x^2-2x+1=c^2-2x+1$ exists.

Condition3: $\lim\limits_{x\to c} f(x)=c^2-2c+1=f(c)$

2 .Discontinuity means a function breaks at a particular point.

Example 34: Discuss the continuity of $f(x)$ if

$$f(x)=\begin{cases}\dfrac{x^3+27}{x+3}; & x\neq -3\\ 27; & x=-3\end{cases}$$

Solution:Condition 1: f(-3)=27, therefore f(x) is defined at x=3

Condition 2: $\lim\limits_{x\to -3}\dfrac{x^3+27}{x+3}=\lim\limits_{x\to -3}\dfrac{(x+3)(x^2-3x+9)}{(x+3)}$

$=\lim\limits_{x\to -3} x^2-3x+9$

$=9+9+9$

$=27$

Condition 3: $\lim\limits_{x\to -3} f(x)=f(-3)=27$

$\therefore f(x)$ is continous.

Example 35: Determine whether or not the function below is continous at x=1

$$f(x)=\begin{cases}\dfrac{x^2-1}{x-1}\ if\ x\neq 1\\ 2 \quad if\ x=1\end{cases}$$

Solution:

Condition 1: $f(1)=2$ hence $f(1)$ is defined.

Condition 2: $\lim_{x\to 1} f(x) = \lim_{x\to 1}\dfrac{x^2-1}{x-1} = \lim_{x\to 1}\dfrac{(x+1)(x-1)}{(x-1)} = 2$ Therefore $\lim_{x\to 1} f(x)$ exists.

Condition3: $\lim f(x) \lim_{x\to 1}\dfrac{x^2-1}{x-1} = f(1)$, hence $f(x)$ is continous at x=1

Example 36: .Discuss the continuity of $f(x)$ if

$$f(x)=\begin{cases}\dfrac{x^2-4}{x-2}, x\neq 2\\ 3 \quad x=2\end{cases}$$

Solution:

Condition 1: f(2) = 3, so f(x) is defined at x=2

Condition 2:

$$\begin{aligned}\lim_{x\to 2} f(x) &= \lim_{x\to 2}\frac{x^2-4}{x-2}\\ &= \lim_{x\to 2}\frac{(x+2)(x-2)}{(x-2)} \quad \text{hence } \lim f(x) \text{ exists.}\\ &= 2\end{aligned}$$

$\therefore f(x)$ Condition 3: $f(2)=3$ but $\lim_{x\to 2} f(x) = 2$ $\therefore \lim_{x\to 2} f(x) \neq f(2)$ Thus *f(x)* is discontinuous at $x=2$

Exercise

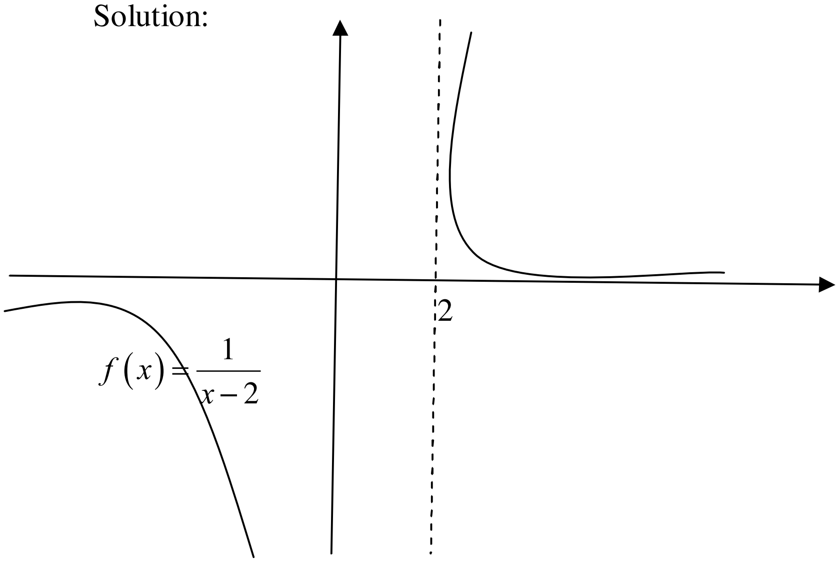
Define the continuity of a real valued function $f(x)$ at a point x=a. Hence determine if the following function is

continous at *x=1*.

$$f(x)=\begin{cases}\dfrac{x^3-1}{x-1}, x\neq 1\\ 3, \quad x=1\end{cases}$$

Example37: Show that $f(x)=\frac{1}{x-2}$ for is $x \neq 2$ is not continous at *x =2*.

Solution:



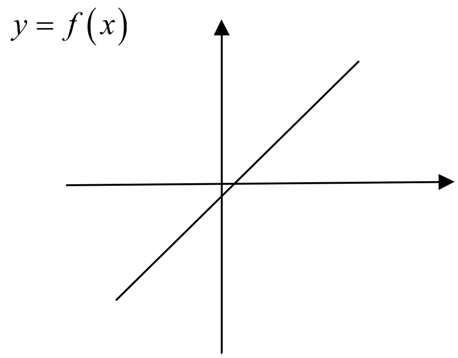
Because f is not defined at the point $x=2$ it is not continous there. Moreover f has what might be called an infinite discontinuity at $x=2$

Combinations of continous Functions.

Any sum or product of continous functions is continous. That is, if the functions f and g are continous at $x=a$, then so are $f+g$ and $f \cdot g$ e.g if f and g are continous at $x=a$, then

$$\lim_{x \to a}\left[f(x)+g(x)\right]=\lim_{x \to a} f(x)+\lim_{x \to a} g(x)=f(a)+g(a)$$

Example 38: $f(x)=x$ is continous everywhere,i.e.



It follows that the cubic polynomial function $f(x)=x^3-3x^2+1$ is continous everywhere. More generally every polynomial function $p(x)=b_n x^n+b_{n-1}x^{n-1}+\ldots+b_1 x+b_0$ is continous at each point of the real line.

If $p(x)$ and $q(x)$ are polynomials, then the quotient law for limits and the continuity of polynomials imply that

$\lim_{x \to a}\frac{p(x)}{q(x)}=\frac{\lim_{x \to a} p(x)}{\lim_{x \to a} q(x)}=\frac{p(a)}{q(a)}$ provided $q(a) \neq 0$. Thus every rational function $f(x)=\frac{p(x)}{q(x)}$ is continous wherever it is defined.

The point $x=a$ where the function f is discontinuous is called a removable discontinuity of f provided that there exists a function F such that $F(x)=f(x)$ for all $x \neq a$ in the domain of f, and this new function F is continous at $x=a$.

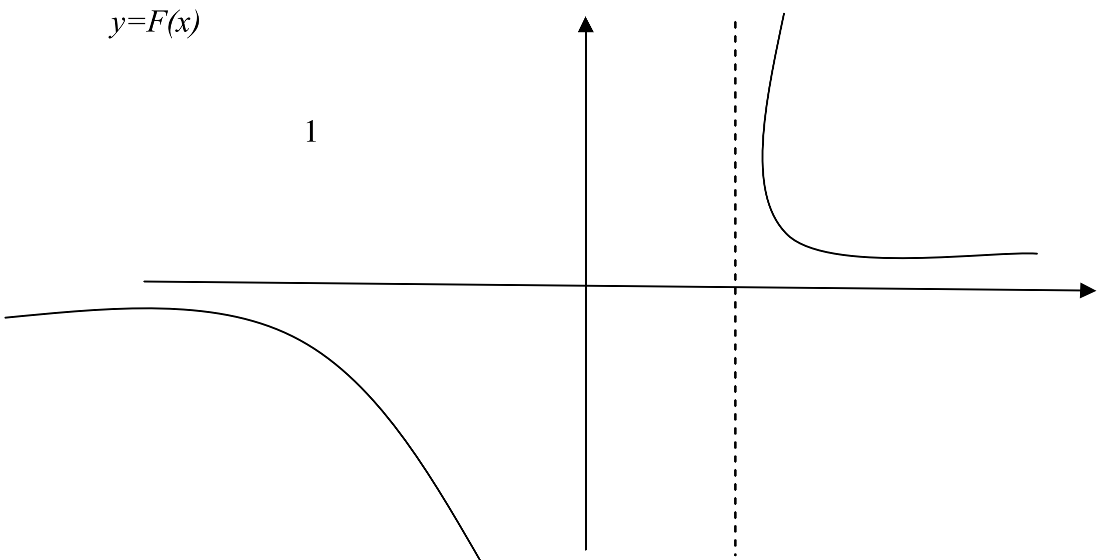
Example 39: Suppose that $f(x)=\dfrac{x-2}{x^2-3x+2}$

$x^2-3x+2=(x-1)(x-2)$

$\therefore f(x)=\dfrac{x-2}{(x-1)(x-2)}$

This shows that f is not defined at $x=1$ and $x=2 \Rightarrow f(x)$ is continous except at these points.

But $f(x)=\dfrac{x-2}{(x-1)(x-2)}=\dfrac{1}{x-1}$. The new function $F(x)=\dfrac{1}{x-1}$ is continous at $x=2$, where $F(2)=1$. Therefore f has a removable discontinuity at $x=2$; the discontinuity at $x=1$ is not removable.



Composition of Continous Functions

Let $h(x)=f(g(x))$ where f and g are continous functions.The composition of two continous functions is continous or more precisely, if g is continous at a and f is continous at $g(a)$, then $f \circ g$ is continous at a where $f \circ g=f(g(x))$.

<u>Proof:</u> The continuity of g at a means that $\lim_{x \to a} g(x)=g(a)$,and the continuity of f at $g(a)$ implies that $\lim_{g(x) \to g(a)} f(g(x))=f(g(a))$

i.e. $\lim_{x \to a} f(g(x))=f\left(\lim_{x \to a} g(x)\right)=f(g(a))$

Example 40: Show that the function $f(x)=\left(\dfrac{x-7}{x^2+2x+2}\right)^{\frac{2}{3}}$ is continous on the whole real line.

Solution: Consider the denominator

$x^2+2x+2=(x+1)^2+1>0$ for all value of x. Hence the rational function

$r(x)=\dfrac{x-7}{x^2+2x+2}$ is defined and continous everywhere. Thus $f(x)=\left(\left[r(x)\right]^2\right)^{\frac{1}{3}}$ is continous everywhere.

One-sided limits

Let $S\subseteq\mathbb{R}$ and $f:S\rightarrow\mathbb{R}$ be a function. If for every $x\in S$, $f(x)\rightarrow L$ as $x\rightarrow a$ and $x>a$ always, then we say that $x\rightarrow a$ from the right and write $x\rightarrow a^{+}$ or we say $\lim\limits_{x\rightarrow a^{+}}f(x)=L$.

Similarly, if $f(x)\rightarrow L$ as $x\rightarrow a$ and $x<a$ always, we say that $x\rightarrow a$ from the left and write $x\rightarrow a^{-}$ or we say $\lim\limits_{x\rightarrow a^{-}}=L$.

The limits $\lim\limits_{x\rightarrow a^{+}}f(x)$ and $\lim\limits_{x\rightarrow a^{-}}f(x)$ are called one-sided limits of f and a

Remarks

1. $\lim\limits_{x\rightarrow a}f(x)=L$ iff $\lim\limits_{x\rightarrow a^{+}}f(x)=\lim\limits_{x\rightarrow a^{-}}f(x)=L$

i.e the limit of a function $f(x)$ exists if the right hand side limit = left-hand side limit.

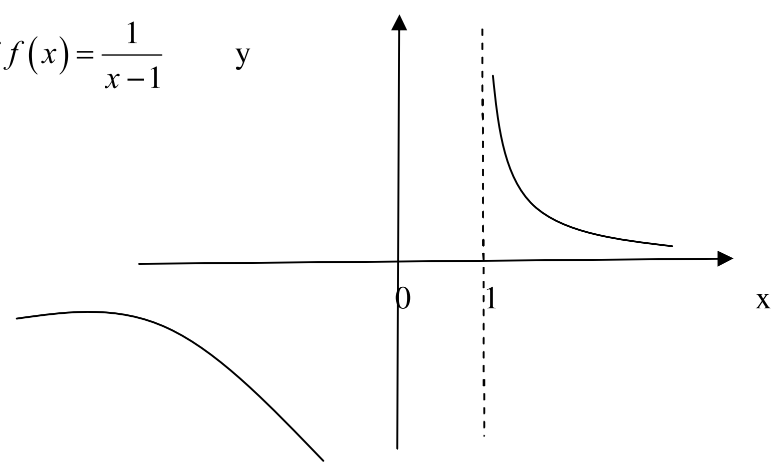
2. If $\lim\limits_{x\rightarrow a^{+}}\neq\lim\limits_{x\rightarrow a^{-}}f(x)$, then $\lim\limits_{x\rightarrow a}f(x)$ does not exist.

Example 41: Given $f(x)=\dfrac{x}{x-1}$,Find $\lim\limits_{x\rightarrow 1^{+}}f(x)$ and $\lim\limits_{x\rightarrow 1^{-}}f(x)$

Solution:

0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2
0	-1	∞	3	2

Also consider the graph of $f(x)=\dfrac{1}{x-1}$



$\lim\limits_{x\to 1^-} f(x) = -\infty \; if \;\; x < 1$

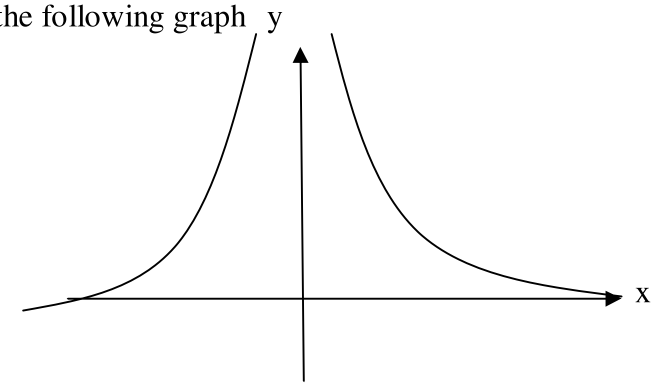
$\lim\limits_{x\to 1^+} f(x) - \infty^1 \; if \;\; x > 1$

$\therefore \lim\limits_{x\to 1} f(x) = \infty \;\Rightarrow \lim\limits_{x\to 1} f(x)$ does not exist.

Example 42: Consider the following graph

$y = f(x) = \dfrac{1}{x^2}$

$y = \dfrac{1}{x^2}$

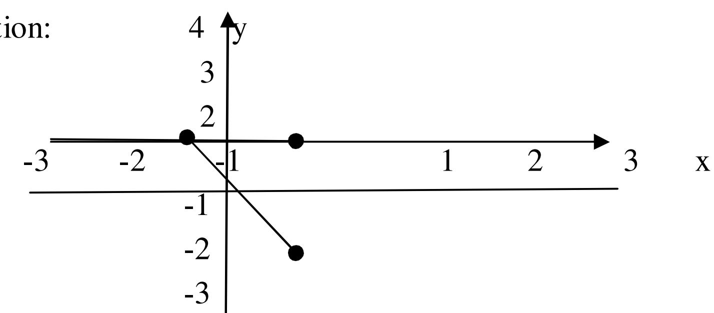


$\lim\limits_{x\to 0^+} f(x) = \infty \qquad \lim\limits_{x\to 0^+} f(x) = \infty$

Example 43: Draw the graph of

$$f(x) = \begin{cases} 1, & if \;\; x = 1 \\ -x, & if \;\; -1 < x < 1 \\ -1, & if \;\; x > 1 \end{cases}$$

Solution:



Example 44: Evaluate $\lim\limits_{x\to 2^+}\left(1+\sqrt{x-2}\right)$ and $\lim\limits_{x\to 2^-}\left(1+\sqrt{x-2}\right)$

Solution:

$\lim\limits_{x\to 2^+}\left(1+\sqrt{x-2}\right)$

$= \lim\limits_{x\to 2^+} 1 + \lim\limits_{x\to 2^+}\sqrt{x-2}$

$= 1 + 0 = 1$

On the other hand, $\lim\limits_{x\to 2^-}\left(1+\sqrt{x-2}\right)$ does not exist (is not real).

Definition: A function f is said to be continous from the right at $x = p$ if $\lim\limits_{x \to p^+} f(x) = f(p)$.

We say that f is continous from the left at q if $\lim\limits_{x \to p^-} f(x) = f(q)$

A function is said to be continous if its continous from the right and from the left i.e

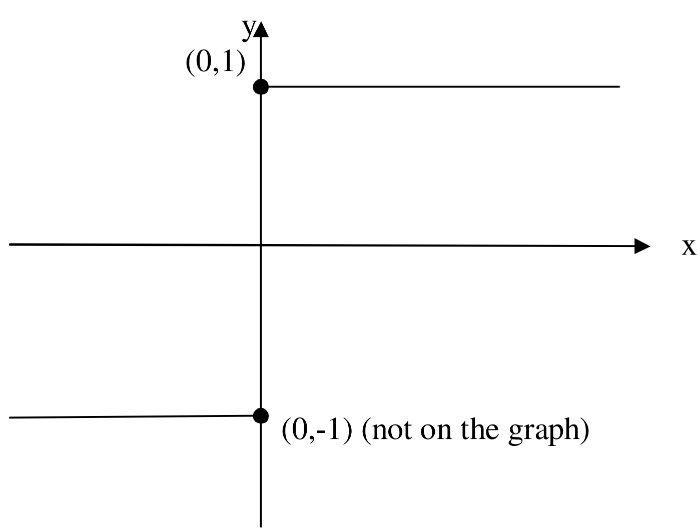
$\lim\limits_{x \to p^+} f(x) = \lim\limits_{x \to p^-} f(x) = f(p)$

Example 45: Discuss the continuity of $g(x) = \sin x = \begin{cases} +1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$

Solution:

$\lim\limits_{x \to 0^-} g(x) = -1$ and $\lim\limits_{x \to 0^+} g(x) = -1$ .Therefore Its left-hand and right-hand limits at $x = 0$ are unequal

Thus $g(x)$ has no limit as $x \to 0$. Hence the function g is not continous at $x = 0$, it has what might be called a finite jump discontinuity there. (see the graph below)



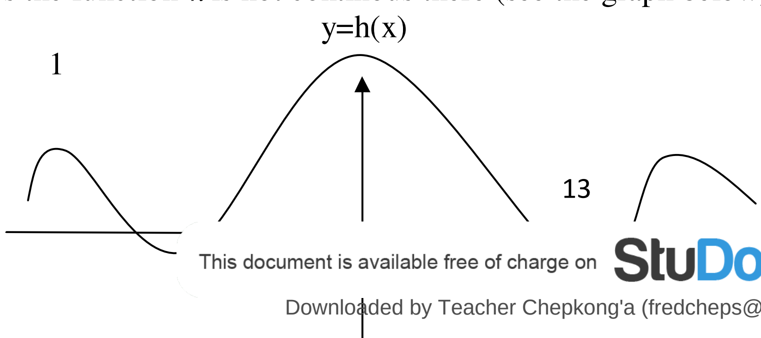
Example 46: Discuss the continuity of $h(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Solution:

$\lim\limits_{x \to 0} h(x) = \lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$ whereas $h(0) = 0$

$\Rightarrow$ the limit and the value of h at $x = 0$ are not equal.

Thus the function h is not continous there (see the graph below)



The point (0,0)is on the graph, the point (0,1) is not.

Remark:
Another way of finding out if functions are continous at $x = a$ is by:

1. Checking if $f(a)$ is defined.
2. Checking if $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$ and exist and are equal.
3. Ensuring that both are equal to $f(a)$.

Example 47: Find the value of c such that $f(x) = \begin{cases} x + c & \text{if } x < 0 \\ 4 - x^2 & \text{if } x \geq 0 \end{cases}$ is continous at $x = 0$.

Solution:
Condition 1: Is $f(x)$ defined at $x = 0$?

Yes, $f(x) = 4 - x^2$
$\therefore f(0) = 4 - 0^2 = 4$

Condition 2: Does $\lim_{x \to 0} f(x)$ exist? In other words,

Does $\lim_{x \to 0^-} f(x)$ exist?

Yes, $\lim_{x \to 0^-} f(x) = 0 + c = c$

Does $\lim_{x \to 0^+}$ exist?

Yes, $\lim_{x \to 0^+} f(x) = 4 - x^2 = 4 - 0^2 = 4$

(c) Is $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$.

For them to be equal, $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) \Rightarrow c = 4$

Thus, $\lim_{x \to 0} f(x)$ exists, i.e. $\lim_{x \to 0} f(x) = 4$

Condition 3: Is $\lim_{x \to 0} f(x) = f(0)$

Yes, $\lim_{x \to 0} f(x) = f(0) = 4$

Conclusion: for $f(x)$ to be continous at $x = 0$, then $c = 4$.

Exercise: Evaluate

1. $\lim_{x \to 3} \frac{x^2 - 2x}{x + 1}$ 2. $\lim_{x \to -\infty} \frac{x - 2}{x^2 + 2x + 1}$ 3. $\lim_{x \to 6^+} \frac{y + 6}{y^2 - 36}$ 4. $\lim_{x \to +\infty} \frac{2 - y}{\sqrt{7 + 6y^2}}$ 5. $\lim_{x \to 3^+} \frac{x}{x - 3}$ 6. $\lim_{x \to 2^-} \frac{x}{x^2 - 4}$

7. $\lim_{y\to 4}\frac{4-y}{2-\sqrt{y}}$ 8. $\lim_{x\to\infty}\sqrt{\frac{5x^2-2}{x+3}}$ 9. $\lim_{x\to\pi}\sin(\frac{x^2}{\pi+x})$

10. For the following problems find the points where given function is not defined and therefore not continous. For each such point *a*, tell whether this discontinuity is removable.

a) $f(x)=\frac{x}{(x+3)^3}$

b) $f(x)=\frac{x-2}{x^2-4}$

c) $f(x)=\frac{1}{1-|x|}$ d) $f(x)=\frac{x-17}{|x-17|}$

e) $f(x)=\begin{cases} -x & \text{if } x<0 \\ x^2 & \text{if } x>0 \end{cases}$ f) $f(x)=\begin{cases} 1+x^2 & \text{if } x<0 \\ \frac{\sin x}{x} & \text{if } x>0 \end{cases}$

11. For the following problems find a value of the constant *c* so that the function $f(x)$ is continous for all *x*.

a) $f(x)=\begin{cases} x+c & \text{if } x<0 \\ 4-x^2 & \text{if } x\geq 0 \end{cases}$ Answer: c=4 b) $f(x)=\begin{cases} 2x+c & \text{if } x\leq 3 \\ 2c-x & \text{if } x>3 \end{cases}$ Answer: c=9

c) $f(x)=\begin{cases} c^2-x^2 & \text{if } x<0 \\ 2(x-c)^2 & \text{if } x\geq 0 \end{cases}$ Answer: c=0 d) $f(x)=\begin{cases} c^3-x^3 & \text{if } x\leq\pi \\ c\sin x & \text{if } x>\pi \end{cases}$

Answer: $c=\pi$