



Pdf SMA 104 Lecture 8(Parametric Differentiation)

Project Management (University of Nairobi)

PARAMETRIC EQUATIONS

Consider $x = f(t)$ and $y = g(t)$, then x and y are both functions of (t) . These equations are called *parametric equations* for x and y and the variable t is called a *parameter*.

Example of parametric equation is $x = 2t$, $y = t^2 - 1$

Derivative of parametric equations

$$x = f(t) \text{ and } y = g(t); \quad \frac{dx}{dt} = f'(t); \quad \frac{dy}{dt} = g'(t); \quad \therefore \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} \text{ where } \frac{dx}{dt} \neq 0$$

Examples

- Find the derivative $\left(\frac{dy}{dx}\right)$ of $x = 2t$, $y = t^2 - 1$

Solution:

$$\frac{dx}{dt} = 2, \quad \frac{dy}{dt} = 2t; \quad \therefore \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = 2t \times \frac{1}{2} = t; \quad \frac{dy}{dx} = t.$$

$$2. \quad x = t^3 + t^2, \quad y = t^2 + t$$

Solution:

$$\frac{dy}{dt} = 3t^2 + 2t, \quad \frac{dy}{dt} = 2t + 1; \quad \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = (2t + 1) \times \frac{1}{3t^2 + 2t} = \frac{2t + 1}{3t^2 + 2t}$$

$$3. \quad x = \cos t, \quad y = \sin t$$

Solution:

$$\frac{dx}{dt} = -\sin t, \quad \frac{dy}{dt} = \cos t; \quad \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{\cos t}{-\sin t} = -\cot t; \quad \frac{dy}{dx} = -\cot t = -\frac{\cos t}{\sin t} = -\frac{x}{y}$$

- Find the gradient of the curve $x = \frac{t}{1+t}$, $y = \frac{t^3}{1+t}$ at the point $\left(\frac{1}{2}, \frac{1}{2}\right)$.

Solution:

$$\frac{dx}{dt} = \frac{(1+t) - t(1)}{(1+t)^2} = \frac{1}{(1+t)^2}; \quad \frac{dy}{dt} = \frac{(1+t)3t^2 - t^3}{(1+t)^2} = \frac{3t^2 + 3t^3 - t^3}{(1+t)^2} = \frac{3t^2 + 2t^3}{(1+t)^2}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{3t^2 + 2t^3}{(1+t)^2} \times (1+t)^2; \quad = 3t^2 + 2t^3$$

$$\text{Where } x = \frac{1}{2}, \quad \frac{t}{1+t} = \frac{1}{2}; \quad 2t = 1+t; \quad t = 1;$$

$$\text{When } y = \frac{1}{2}; \quad \frac{t^3}{1+t} = \frac{1}{2}; \quad 2t^3 = 1+t; \quad t = 1;$$

$$\text{When } t = 1, \quad \frac{dy}{dx} = 3(1^2) + 2(1^3) = 5$$

- If $x = t^3 + t^2$, $y = t^2 + t$ find $\frac{dy}{dx}$ in terms of t .

Solution:

$$\frac{dx}{dt} = 3t^2 + 2t, \quad \frac{dy}{dt} = 2t + 1; \quad \therefore \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = 2t + 1 \times \frac{1}{3t^2 + 2t} = \frac{2t+1}{3t^2+2t} = \frac{2t+1}{t(3t+2)}$$

6. If $x = \frac{2t}{t+2}$, $y = \frac{3t}{t+3}$, find $\frac{dy}{dx}$ in terms of t .

Solution:

$$\frac{dx}{dt} = \frac{(t+2)2 - 2t(1)}{(t+2)^2} = \frac{2t+4-2t}{(t+2)^2} = \frac{4}{(t+2)^2}$$

$$\frac{dy}{dt} = \frac{(t+3)3 - 3t(1)}{(t+3)^2} = \frac{3t+9-3t}{(t+3)^2} = \frac{9}{(t+3)^2}$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{9}{(t+3)^2} \times \frac{(t+2)^2}{4} = \frac{9(t+2)^2}{4(t+3)^2}$$

Exercise

1. Find $\frac{dy}{dx}$, in terms of t when a) $x = at^2$, $y = 2at$; b) $x = (t+1)^2$, $y = (t^2 - 1)$;
2. $x = \cos^2 t$, $y = \sin^2 t$; 3. $x = t$, $y = \frac{1}{t}$; 4. $x = t^2 - \frac{\pi}{2}$, $y = \sin(t^2)$;

Parametric formula for $\frac{d^2y}{dx^2}$

Let $x = f(t)$, $y = g(t)$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} \quad \left(\frac{dx}{dt} \neq 0 \right) = y'; \quad \frac{d^2y}{dx^2} = \frac{dy'}{dx} \quad \text{But by chain rule}$$

$$\frac{dy'}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}; \quad \therefore \frac{d^2y}{dx^2} = \frac{dy}{dt} \times \frac{dt}{dx}$$

Similarly,

$$\frac{d^3y}{dx^3} = \frac{d}{dx} \left(\frac{dy}{dx} \right); \quad \frac{d}{dx} \left(\frac{dy}{dx} \right) \times \frac{dt}{dx}$$

Example

1. Find $\frac{d^2y}{dx^2}$ if $x = t - t^2$, $y = t - t^3$

Solution:

$$\frac{dx}{dt} = 1 - 2t, \quad \frac{dy}{dt} = 1 - 3t^2; \quad \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{1-3t^2}{1-2t}; \quad y' = \frac{1-3t^2}{1-2t}$$

$$\frac{dy'}{dt} = \frac{(1-2t)(-6t) - (1-3t^2)(-2)}{(1-2t)^2} = \frac{-6t+12t^2+2-6t^2}{(1-2t)^2} \quad \therefore \frac{dy'}{dt} = \frac{6t^2-6t+2}{(1-2t)^2}$$

$$\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{dy'}{dt} \times \frac{dt}{dx} = \frac{6t^2 - 6t + 2}{(1-2t)^2} \times \frac{1}{1-2t} = \frac{6t^2 - 6t + 2}{(1-2t)^3}$$

2. Find $\frac{d^2y}{dx^2}$ where $x = \cos^2 t$, $y = \sin^2 t$.

Solution:

$$\frac{dx}{dt} = 2(-\sin t)\cos t; \quad \frac{dy}{dt} = 2(\cos t)\sin t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{2\cos t \sin t}{-2\sin t \cos t} = -1 \therefore \frac{d^2y}{dx^2} = 0$$

Find $\frac{d^2y}{dx^2}$ if $x = t$, $y = \sqrt{t}$;

Solution:

$$\frac{dx}{dt} = 1; \quad \frac{dy}{dt} = \frac{1}{2}t^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{1}{2}t^{-\frac{1}{2}} \times \frac{1}{1} = \frac{1}{2\sqrt{t}}; \quad \frac{dy'}{dt} = -\frac{1}{4}t^{-\frac{3}{2}}$$

$$\frac{d^2y}{dx^2} = \frac{dy'}{dt} \times \frac{dt}{dx} = -\frac{1}{4}t^{-\frac{3}{2}} = -\frac{1}{4t^{\frac{3}{2}}} = -\frac{1}{4(\sqrt{t})^3}$$

$$4. y = \sin t, \quad x = \cos t + \ln\left(\tan\left(\frac{1}{2}t\right)\right)$$

Solution:

$$\frac{dy}{dt} = \cos t; \quad \frac{dx}{dt} = -\sin t + \frac{1}{2}\sec^2 \frac{1}{2}t = -\sin t + \left(\frac{1}{2\cos^2 \frac{1}{2}t} \times \frac{\cos \frac{1}{2}t}{\sin \frac{1}{2}t}\right) = -\sin t + \frac{1}{2\cos \frac{1}{2}t \sin \frac{1}{2}t}$$

$$= -\sin t + \frac{1}{\sin t}$$

$$\frac{dx}{dt} = \frac{-\sin^2 t + 1}{\sin t} = \frac{\cos^2 t}{\sin t}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \cos t \times \frac{\sin t}{\cos^2 t} = \tan t$$

$$\frac{d^2y}{dx^2} = \sec^2 t \times \frac{\sin t}{\cos^2 t} = \frac{\sin t}{\cos^4 t} = \tan t \sec^3 t$$

Exercise

Find $\frac{d^2y}{dx^2}$ where 1. $x = 2t - 5$, $y = 4t - 7$; 2. $x = \cos t$, $y = 5\sin t$; 3.

$$x = t^2 - \frac{\pi}{2}, \quad y = \sin(t^2)$$

HIGHER DERIVATIVES

If f is a differentiable function, then its derivative f' is also a function, so f' may have a derivative of its own, denoted by $(f')' = f''$. This new function f'' is called the second derivative of f because it is the derivative of the derivative of f . Thus

$$f''(x) = \frac{d}{dx}(f'(x)) = \frac{d}{dx}\left(\frac{d}{dx}f(x)\right)$$

Examples

$$1. \text{ If } f(x) = x^8, \text{ then } f'(x) = 8x^7. \text{ So } f''(x) = \frac{d}{dx}(f'(x)) = \frac{d}{dx}(8x^7) = 56x^6$$

$$\text{Notation: If } y = f(x), \text{ then } y'' = f''(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

$$y''' = f'''(x) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$$

The process can be continued. The fourth derivative f''' is usually denoted by $f^{(iv)}$ or $f^{(4)}$. In general, the n^{th} derivative of f is denoted by $f^{(n)}$ and is obtained from f by differentiating n times. If $y = f(x)$, we write

$$y^n = f^n = \frac{d^n y}{dx^n}$$

$$2. \text{ If } y = x^3 - 6x^2 - 5x + 3; \quad y' = 3x^2 - 12x - 5; \quad y'' = 3x^2 - 12x - 12; \quad y''' = 6, \quad y^{(iv)} = 0$$

$$3. \text{ Find } y'' \text{ if } x^4 + y^4 = 16$$

Solution: Differentiating implicitly w.r.t. x ,

$$\begin{aligned} 4x^3 + 4y^3 \frac{dy}{dx} &= 0; \quad 4y^3 \frac{dy}{dx} = -4x^3; \quad \frac{dy}{dx} = -\frac{x^3}{y^3} = -\left(\frac{x}{y}\right)^3; \quad \frac{dy}{dx} = -\frac{x^3}{y^3} \\ \frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(-\frac{x^3}{y^3}\right) = \frac{d}{dx}(-x^3y^{-3}) = -x^3(-3)y^{-4} \frac{dy}{dx} + y^{-3}(-3x^2) = \frac{3x^3}{y^4} \frac{dy}{dx} - \frac{3x^2}{y^3} \\ \frac{d^2y}{dx^2} &= \frac{3x^3}{y^4} \times \left(-\frac{x^3}{y^3}\right) - \frac{3x^2}{y^3} = \frac{-3x^6}{y^7} - \frac{3x^2}{y^3} = \frac{-3x^6 - 3x^2y^4}{y^7} = \frac{-3x^2(x^4 + y^4)}{y^7} \quad \text{but} \end{aligned}$$

$$y^4 + x^4 = 16$$

$$\frac{d^2y}{dx^2} = -\frac{48x^2}{y^7}$$

Or

$$12x^2 + 4\left(3y^2\left(\frac{dy}{dx}\right)^2 + y^3 \frac{d^2y}{dx^2}\right) = 0$$

$$12x^2 + 12y^2 \left(\frac{dy}{dx} \right)^2 + 4y^3 \frac{d^2y}{dx^2} = 0$$

$$3x^2 + 3y^2 \left(\frac{dy}{dx} \right)^2 + y^3 \frac{d^2y}{dx^2} = 0$$

$$\frac{d^2y}{dx^2} = \frac{-3x^2}{y^3} - \frac{3y^2}{y^3} \left(\frac{dy}{dx} \right)^2 = -\frac{3x^2}{y^3} - \frac{3}{y} \left(\frac{x^6}{y^6} \right)$$

$$= -\frac{3x^2}{y^3} - \frac{3x^6}{y^7} = \frac{-3x^2 y^4}{y^7} - \frac{3x^6}{y^7} = \frac{-3x^2(y^4 + x^4)}{y^7} = \frac{-3x^2(16)}{y^7} = \frac{-48x^2}{y^7}$$

Examples

1. If $4x^2 - 2y^2 = 9$, find $\frac{d^2y}{dx^2}$

Solution:

$$8x - 4y \frac{dy}{dx} = 0; \quad \frac{dy}{dx} = \frac{8x}{4y} = \frac{2x}{y};$$

$$8 - 4 \left\{ y \frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot \frac{dy}{dx} \right\} = 0$$

$$8 - 4y \frac{d^2y}{dx^2} - 4 \left(\frac{dy}{dx} \right)^2 = 0$$

$$-4y \frac{d^2y}{dx^2} = 4 \left(\frac{dy}{dx} \right)^2 - 8$$

$$\frac{d^2y}{dx^2} = \frac{8 - 4 \left(\frac{dy}{dx} \right)^2}{4y}; \text{ But } \frac{dy}{dx} = \frac{2x}{y},$$

$$\frac{d^2y}{dx^2} = \frac{8 - 4 \left(\frac{2x}{y} \right)^2}{4y} = \frac{8 - \frac{16x^2}{y^2}}{4y} = \frac{8y^2 - 16x^2}{4y^3} = \frac{2y^2 - 4x^2}{y^3}$$

$$\text{Or } \frac{dy}{dx} = \frac{2x}{y},$$

$$\frac{d^2y}{dx^2} = \frac{2y - 2x \cdot \frac{dy}{dx}}{y^2} = \frac{2y - 2x \left(\frac{2x}{y} \right)}{y^2} = \frac{2y - \frac{4x^2}{y}}{y^2} = \frac{2y^2 - 4x^2}{y^3}$$

$$2x^2 + 3xy - y^2 = 3; \text{ Find } \frac{d^2y}{dx^2} \text{ at } (1,1).$$

Solution:

$$\begin{aligned} & 2x + 3\left[x \frac{dy}{dx} + y\right] - 2y \frac{dy}{dx} = 0 \\ & 2x + 3x \frac{dy}{dx} + 3y - 2y \frac{dy}{dx} = 0; \quad \frac{dy}{dx} = \frac{-2x - 3y}{3x - 2y} \\ & 2 + 3\left[x \frac{d^2y}{dx^2} + \frac{dy}{dx}\right] + 3 \frac{dy}{dx} - 2\left[y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] = 0 \\ & \frac{d^2y}{dx^2} = \frac{2\left(\frac{dy}{dx}\right)^2 - 6\frac{dy}{dx} - 2}{3x - 2y}; \quad \text{But } \frac{dy}{dx} = \frac{-2x - 3y}{3x - 2y} = \frac{dy}{dx} = \frac{-2(1) - 3(1)}{3(1) - 2(1)} = \frac{-5}{1} = -5 \\ & \frac{d^2y}{dx^2} = \frac{2(-5)^2 - 6(-5) - 2}{3(1) - 2(1)} = 50 + 30 - 2 = 78 \end{aligned}$$

$$3. \quad 3x^2 + 5xy + 4y^2 - 4y = 0. \text{ Find } \frac{d^2y}{dx^2} \text{ at } (0,1).$$

Solution:

$$\begin{aligned} & 6x + 5(x \frac{dy}{dx} + \frac{dy}{dx}) + 8y \frac{dy}{dx} - 4 \frac{dy}{dx} = 0 \\ & 6x + 5x \frac{dy}{dx} + 5 \frac{dy}{dx} + 8y \frac{dy}{dx} - 4 \frac{dy}{dx} = 0 \\ & 6 + 5(x \frac{d^2y}{dx^2} + \frac{dy}{dx}) + 5 \frac{d^2y}{dx^2} + 8[y \frac{dy}{dx} + (\frac{dy}{dx})^2] - 4 \frac{d^2y}{dx^2} = 0 \\ & 5x \frac{d^2y}{dx^2} + 5 \frac{d^2y}{dx^2} - 4 \frac{d^2y}{dx^2} = -6 - 5 \frac{dy}{dx} - 8y \frac{dy}{dx} - 8(\frac{dy}{dx})^2 \\ & 5x \frac{d^2y}{dx^2} + \frac{d^2y}{dx^2} = -6 - 5 \frac{dy}{dx} - 8y \frac{dy}{dx} - 8(\frac{dy}{dx})^2 \\ & \frac{d^2y}{dx^2} = \frac{-6 - 5 \frac{dy}{dx} - 8y \frac{dy}{dx} - 8(\frac{dy}{dx})^2}{5x + 1} \end{aligned}$$

But

$$5x \frac{dy}{dx} + 5 \frac{dy}{dx} + 8y \frac{dy}{dx} - 4 \frac{dy}{dx} = -6x \Rightarrow \frac{dy}{dx} = \frac{-6x}{5x + 5 + 8y - 4} = \frac{dy}{dx} = \frac{-6x}{5x + 8y + 1} = \frac{-6(0)}{5(0) + 8(1) + 1} = 0$$

$$\frac{d^2y}{dx^2} = \frac{-6 - 5(0) - 8(1)(0) - 8(0)^2}{5(0) + 1} = 0$$

Exercise

1. Find y' , y'' , y''' where a) $y = \frac{x}{1-x}$ b) $y = \sqrt{x^2 + 1}$ c) $x^3 + y^3 = 1$ d) $x^2 + 6xy + y^2 = 8$ e) $\sqrt{x} + \sqrt{y} = 1$
2. Let $9y = x^3 + 3x + 1$. Show that $y''' + xy'' - 2y' = 0$
3. Let $y = \frac{1}{x}$ ($x \neq 0$). Show that $x^3y'' + x^2y' - xy = 0$