



Pdf SMA 104 Lecture 9(Tangents, Normals, Small Changes)

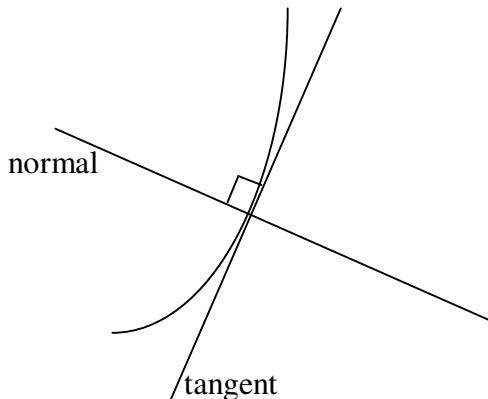
Project Management (University of Nairobi)

APPLICATIONS OF DIFFERENTIATION

EQUATIONS OF TANGENTS AND NORMALS

Definition: A normal to a curve at a point is the straight line through the point at right angles to the tangent at the point.

$$y=f(x)$$



Finding the equations of tangents and normals.

Examples

- Find the equation of the tangent to the curve $y = x^3$ at the point (2,8).

Solution:

$$y = x^3; \therefore \text{gradient of } y \text{ or } \frac{dy}{dx} = 3x^2$$

$$\text{When } x=2; \frac{dy}{dx} = 3 \times 2^2 = 3 \times 4 = 12$$

Thus the gradient of the tangent at (2, 8) is 12.

$$\text{But gradient} = \frac{\Delta y}{\Delta x} = \frac{y - 8}{x - 2} = 12;$$

$$y - 8 = 12(x - 2)$$

$$y - 8 = 12x - 24$$

$$y = 12x - 24 + 8$$

$$y = 12x - 16 \text{ which is the equation of the tangent.}$$

- Find the equation of the normal to the curve $y = (x^2 + x + 1)(x - 3)$ at the point where it cuts the x -axis.

Solution:

$$y = (x^2 + x + 1)(x - 3). \text{ At the } x\text{-axis } y=0.$$

$$\text{When } y=0, (x^2 + x + 1)(x - 3) = 0; x - 3 = 0; x = 3.$$

$$\text{Or } x^2 + x + 1 = 0; x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1-4}}{2} (\text{No real roots}).$$

Hence $x = 3$ and therefore the curve cuts the x -axis at (3,0).

$$\therefore \text{gradient} \left(\frac{dy}{dx} \right) = (2x+1)(x-3) + 1(x^2 + x + 1) = 2x^2 - 6x + x - 3 + x^2 + x + 1$$

$$\frac{dy}{dx} = 3x^2 - 4x - 2; \text{ When } x=3, \frac{dy}{dx} = 3(3^2) - 4(3) - 2 = 27 - 12 - 2 = 13$$

The gradient of the tangent at (3,0) is 13, \therefore the gradient of the normal at (3,0) is $-\frac{1}{13}$ (since for perpendicular lines with gradients m_1 and m_2 , $m_1 \times m_2 = -1$)

$$\frac{\Delta y}{\Delta x} = \frac{y-0}{x-3} = -\frac{1}{13}; 13y = -x + 3; y = -\frac{x}{13} + \frac{3}{13} \text{ is the equation of the normal.}$$

3. Find the slope (gradient) of the tangent to the curve $x^2 + xy + y^2 = 7$ at the point (1,2).

Solution:

$$2x + y + x \frac{dy}{dx} + 2y \frac{dy}{dx} = 0; (x + 2y) \frac{dy}{dx} = -2x - y; \frac{dy}{dx} = \frac{-2x - y}{x + 2y}.$$

$$\text{At } x=1, y=2, \frac{dy}{dx} = \frac{-2(1)-2}{1+2(2)} = \frac{-4}{5}.$$

4. Find the equation of the tangent and a normal to the curve $x^2 + xy - y^2 = 1$ at the point (2, 3).

Solution:

Equation of the tangent

$$2x + y + x \frac{dy}{dx} - 2y \frac{dy}{dx} = 0; (x - 2y) \frac{dy}{dx} = -2x - y; \frac{dy}{dx} = \frac{-2x - y}{x - 2y} = \frac{-2(2) - 3}{2 - 2(3)} = \frac{7}{4} \text{ at the point}$$

$$x=2, y=3.$$

Equation of the normal

Recall: For perpendicular lines with gradients m_1 and m_2 , $m_1 \times m_2 = -1$

$$\therefore \text{The gradient of the normal at the point (2, 3) is } -\frac{4}{7}.$$

$$\frac{\Delta y}{\Delta x} = \frac{y-3}{x-2} = -\frac{4}{7}; 7(y-3) = -4(x-2); 7y - 21 = -4x + 8; 7y = -4x + 8 + 21;$$

$$y = \frac{-4x}{7} + \frac{29}{7} = \frac{1}{7}[29 - 4x]$$

5. Find the normal to a curve $3xy + 2y^2 - x^3 = 0$ at the point (1,2).

Solution:

$$3y + 3x \frac{dy}{dx} + 4y \frac{dy}{dx} - 3x^2 = 0; (3x + 4y) \frac{dy}{dx} = 3x^2 - 3y; \frac{dy}{dx} = \frac{3x^2 - 3y}{3x + 4y}$$

$$\text{At } x=1, y=2, \frac{dy}{dx} = \frac{3-6}{3+8} = \frac{-3}{11}; \therefore \text{Gradient of the normal to the curve at (1,2) is } \frac{11}{3}.$$

$$\frac{\Delta y}{\Delta x} = \frac{y-2}{x-1} = \frac{11}{3}; 3y - 6 = 11x - 11; 3y = 11x - 5; y = \frac{11x}{3} - \frac{5}{3}$$

6. Find the equations of the tangent and the normal to the curve $x^2 + 2xy - y^2 = 4$ at the point $(2,4)$.

Solution:

$$2x + 2y + 2x \frac{dy}{dx} - 2y \frac{dy}{dx} = 0; \quad (2x - 2y) \frac{dy}{dx} = -2x - 2y; \quad \frac{dy}{dx} = \frac{-2x - 2y}{2x - 2y} = \frac{-4 - 8}{4 - 8} = \frac{-12}{-4} = 3 \text{ at } (2,4).$$

\therefore the gradient of the tangent line at $(2,4)$ is 3.

\therefore gradient of the normal to the curve is $-\frac{1}{3}$.

$$\frac{\Delta y}{\Delta x} = \frac{y - 4}{x - 2} = -\frac{1}{3}; \quad 3y - 12 = -x + 2; \quad 3y = -x + 14; \quad y = \frac{1}{3}[14 - x]$$

7. The parametric equations of a curve are $x = t^2 - 4$ and $y = t^3 - 4t$. Find the equation of the tangent to the curve at the point $(-3,3)$.

Solution:

$$\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = 3t^2 - 4; \quad \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{3t^2 - 4}{2t}. \quad \text{But } x = t^2 - 4, \quad y = t^3 - 4t.$$

When $x = -3, \quad -3 = t^2 - 4, \quad t^2 = 1; \quad t = \pm 1$

When $y = 3, \quad 3 = t^3 - 4t; \quad t^3 - 4t - 3 = 0; \quad \text{when } t = -1, \quad -1 + 4 - 3 = 0 \text{ and therefore } (t+1) \text{ is a factor of } t^3 - 4t - 3.$

$$\begin{array}{r} t^2 - t - 3 \\ (t+1) \overline{)t^3 - 4t - 3} \\ t^3 + t^2 \\ \hline 0 - t^2 - 4t \\ -t^2 - t \\ \hline -3t - 3 \\ -3t - 3 \\ \hline 0 \end{array}$$

$$(t+1)(t^2 - t - 3) = 0; \quad t = -1 \text{ or } t^2 - t - 3 = 0; \quad t = \frac{1 \pm \sqrt{1+12}}{2} = \frac{1 \pm \sqrt{13}}{2}$$

$$\text{At } t = -1, \quad \frac{dy}{dx} = \frac{3-4}{2(-1)} = \frac{1}{2};$$

$$\frac{\Delta y}{\Delta x} = \frac{y - 3}{x + 3} = \frac{1}{2}; \quad 2(y - 3) = x + 3; \quad 2y - 6 = x + 3; \quad 2y = x + 9; \quad y = \frac{x}{2} + \frac{9}{2};$$

Exercise

1. Find the equation of a tangent and normal to the curve $x^2 + y^2 - 6xy + 3x - 2y + 5 = 0$ at a point $(3,0)$.

2. Find the equation of the tangent and normal to the curve $x = \frac{t}{1+t}$, $y = \frac{t^3}{1+t}$ at the point $\left(\frac{1}{2}, \frac{1}{2}\right)$.

SMALL CHANGES

Recall:

$\frac{d}{dx} f(x) = \lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x}$ approaches the tangent line.

$$\therefore \text{if } \delta x \text{ is small, then we say that } \frac{\delta y}{\delta x} \approx \frac{dy}{dx} \Rightarrow \delta y \approx \frac{dy}{dx} \cdot \delta x$$

This approximation can be used to estimate the value of a function close to a known value.i.e $y + \delta y$ can be approximated if y is known.

Examples

1. Use $y = \sqrt{x}$ to approximate the value of $\sqrt{1.1}$.

Solution:

Known value $\sqrt{1} = 1$.

From $\sqrt{1.1} = \sqrt{1+0.1}$, $x = 1$, $\delta x = 0.1$

$$y = \sqrt{x}; \quad \frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

$$\text{From } \delta y \approx \frac{dy}{dx} \cdot \delta x \approx \frac{1}{2\sqrt{x}} \cdot \delta x \approx \frac{1}{2\sqrt{1}} \times 0.1 \approx 0.05 \therefore \delta y \approx 0.05.$$

$$\therefore \sqrt{1.1} \approx y + \delta y \approx \sqrt{1} + 0.05; \quad \sqrt{1.1} \approx 1.05$$

2. Approximate $\ln 1.1$

Solution:

Known value $= \ln 1 = 0$

Let $y = \ln x$; $x = 1$, $\delta x = 0.1$

$$\frac{dy}{dx} = \frac{1}{x}; \quad \text{But } \delta y \approx \frac{dy}{dx} \cdot \delta x \approx \frac{1}{x} \cdot \delta x \approx \frac{1}{1} \times 0.1 \approx 0.1$$

$$\therefore \ln(1.1) \approx y + \delta y \approx \ln x + \delta y \approx \ln 1 + \delta y \approx 0 + 0.1 \approx 0.1 \therefore \ln 1.1 \approx 0.1.$$

3. Approximate $\sqrt{101}$.

Solution:

Known value $= 100$

Let $y = \sqrt{x}$, $x = 100$, $\delta x = 1$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}; \quad \text{But } \delta y \approx \frac{dy}{dx} \cdot \delta x \approx \frac{1}{2\sqrt{100}} \times 1 \approx \frac{1}{20}$$

$$\therefore \sqrt{101} \approx y + \delta y \approx \sqrt{x} + \frac{1}{20} \approx \sqrt{100} + \frac{1}{20} = 10 + 0.05; \quad \therefore \sqrt{101} = 10.05.$$

4. By taking $1^0 = 0.0175$ radians, approximate $\sin 29^0$.

Solution:

$$\text{Known value } \sin 30^0 = \frac{1}{2}; \quad \text{Let } y = \sin x; \quad x = 30^0; \quad \delta x = -1$$

$$\begin{aligned}\frac{dy}{dx} &= \cos x; \quad \delta y \approx \frac{dy}{dx} \cdot \delta x = \cos x \cdot (-1^0) \\ &\approx \cos 30^0 \times (-1^0); \quad \text{But } -1^0 = -0.0175 \text{ radians},\end{aligned}$$

$$\delta y = \frac{\sqrt{3}}{2}(-0.0175) \approx -\frac{\sqrt{3}}{2}(0.0175)$$

$$\therefore \sin 29^0 \approx y + \delta y \approx \sin 30 + \delta y \approx \sin 30 + \delta y \approx \frac{1}{2} - 0.015 \approx 0.4848$$

5. Approximate $\sqrt[3]{65}$.

Solution:

$$\text{Known value } = \sqrt[3]{64} = 4$$

$$\text{Let } y = 3\sqrt{x}, \quad x = 64; \quad \delta x = 1$$

$$y = x^{\frac{1}{3}}; \quad \frac{dy}{dx} = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3x^{\frac{2}{3}}}; \quad \delta y \approx \frac{dy}{dx} \times \delta x \approx \frac{1}{3x^{\frac{2}{3}}} \times 1 \approx \frac{1}{3(\sqrt[3]{64})^2} \times 1 = \frac{1}{16 \times 3} = \frac{1}{48};$$

$$\therefore \sqrt[3]{65} = y + \delta y \approx \sqrt[3]{64} + \delta y \approx 4 + \frac{1}{48} \approx 4.021$$

6. The side of a square is 5cm. Find the increase in the area of the square when the side expands by 0.01cm.

Solution:

Let the area of the square be $A \text{ cm}^2$ when the side is $x \text{ cm}$.

Then $A = x^2$.

$$\text{Now, } \delta A \approx \frac{dA}{dx} \delta x \quad x = 5; \quad \delta x = 0.01$$

$$A = x^2; \quad \frac{dA}{dx} = 2x \quad \therefore \delta A \approx 2x(0.01) \approx 2 \times 5(0.01) \approx 0.1$$

\therefore the increase in the area is ≈ 0.1

7. Find approximation for $\sqrt{9.01}$

Solution:

$$\text{Known value } = \sqrt{9} = 3$$

$$\text{Let } y = \sqrt{x}, \quad x = 9; \quad \delta x = 0.01$$

$$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}; \quad \delta y \approx \frac{dy}{dx} \cdot \delta x \approx \frac{1}{2\sqrt{x}} \times 0.01 \approx \frac{1}{2\sqrt{9}} \times 0.01 \approx \frac{1}{6} \times 0.01 = \frac{1}{600}$$

$$\therefore \sqrt{9.01} = y + \delta y \approx \sqrt{9} + \delta y \approx \sqrt{9} + \frac{1}{600} \approx 3 + \frac{1}{600} \approx 3.00167$$

8. Given that $\sin 60^\circ = 0.86605$, $\cos 60^\circ = 0.50000$, and $1^\circ = 0.001745$ radians, Use $\frac{\delta y}{\delta x} \approx \frac{dy}{dx}$ to calculate the value of $\sin 60.1^\circ$ correct to 5.d.p.

Solution:

$$y = \sin x, x = 60^\circ; \delta x = 0.1^\circ$$

$$\begin{aligned}\frac{dy}{dx} &= \cos x; \delta y \approx \frac{dy}{dx} \cdot \delta x \approx \cos x(0.1)^\circ \approx \cos 60^\circ(0.001745) \approx (0.5)(0.001745) \\ &\approx 0.00008725 \therefore \sin(60.1^\circ) \approx y + \delta y \approx \sin x + \delta y \approx \sin 60^\circ + (0.5)(0.001745) \\ &\approx 0.86605 + (0.5)(0.001745) = 0.86613725\end{aligned}$$

MIXED EXERCISE

ATTEMPT ALL QUESTIONS

1. (a) Use the linear approximation formula to approximate $(626)^{\frac{3}{4}}$.

(b) Find $\frac{dy}{dx}$ if (i) $y = \tan^{-1}(x^2 + 1)$ (ii) $y = \sin^{-1} x$

(c) Find $\frac{dy}{dx^2}$ and $\frac{d^2y}{dx^2}$, given $xy + x - 2y - 1 = 0$.

- 2.(a) If $x = \cos t$ and $y = 1 - \sin^2 t$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

(b) Use logarithmic differentiation to evaluate $\frac{dy}{dx}$ if

$$(i) y = \frac{\sin x \cos x \tan^3 x}{\sqrt{x}} \quad (ii) y = \frac{(x^2 + 1)\cot x}{3 - \cot x}$$

- (c) Find the equation of a tangent and normal to the curve $x^2 + y^2 - 6xy + 3x - 2y + 5 = 0$ at the point $(3, 0)$.

- 3.(a) Differentiate $f(x) = \cot x$ from first principles.

(b) Find $\frac{dy}{dx}$ if (i) $y = 4^x$ (ii) $y = \ln(\cot x - \cos ec x)$ (iii) $y = x \sin^{-1}(3x) - \sqrt{1 - 9x^2}$ (iv)

$$y = \ln\left(\frac{1 + \sin x}{1 - \sin x}\right)^{\frac{1}{2}}$$